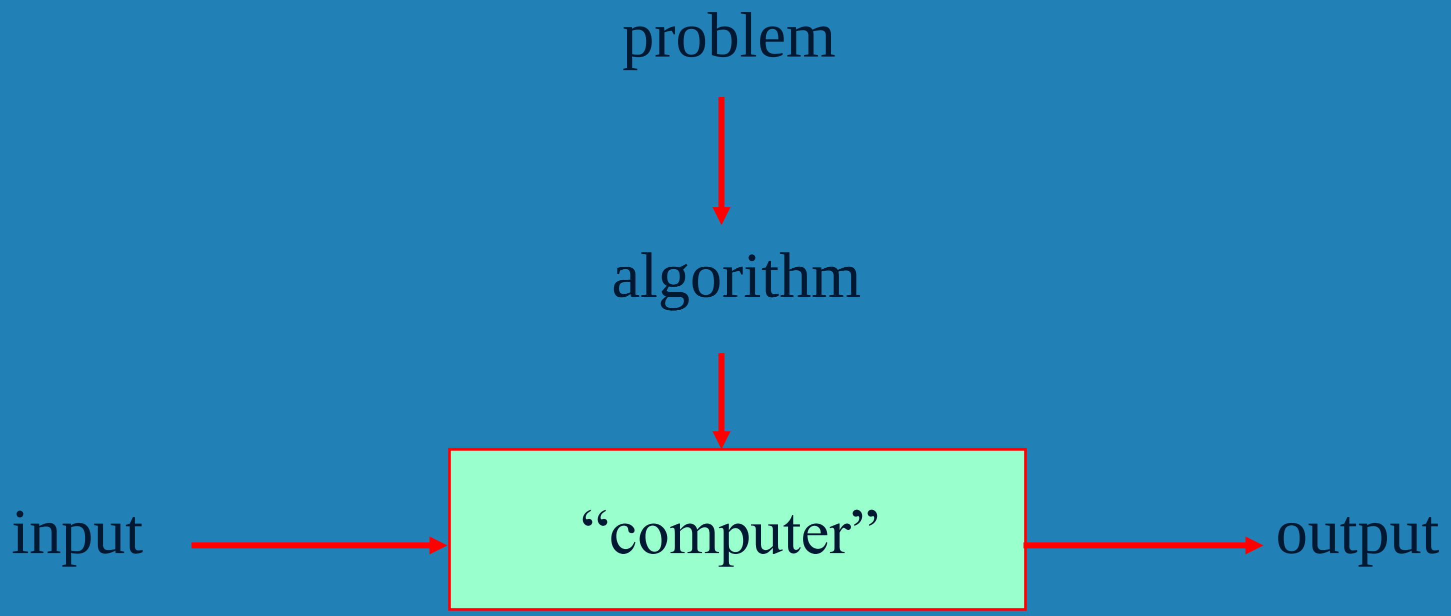
Introduction

What is an algorithm?

An *algorithm* is a sequence of unambiguous instructions for solving a problem, i.e., for obtaining a required output for any legitimate input in a finite amount of time.



Algorithm

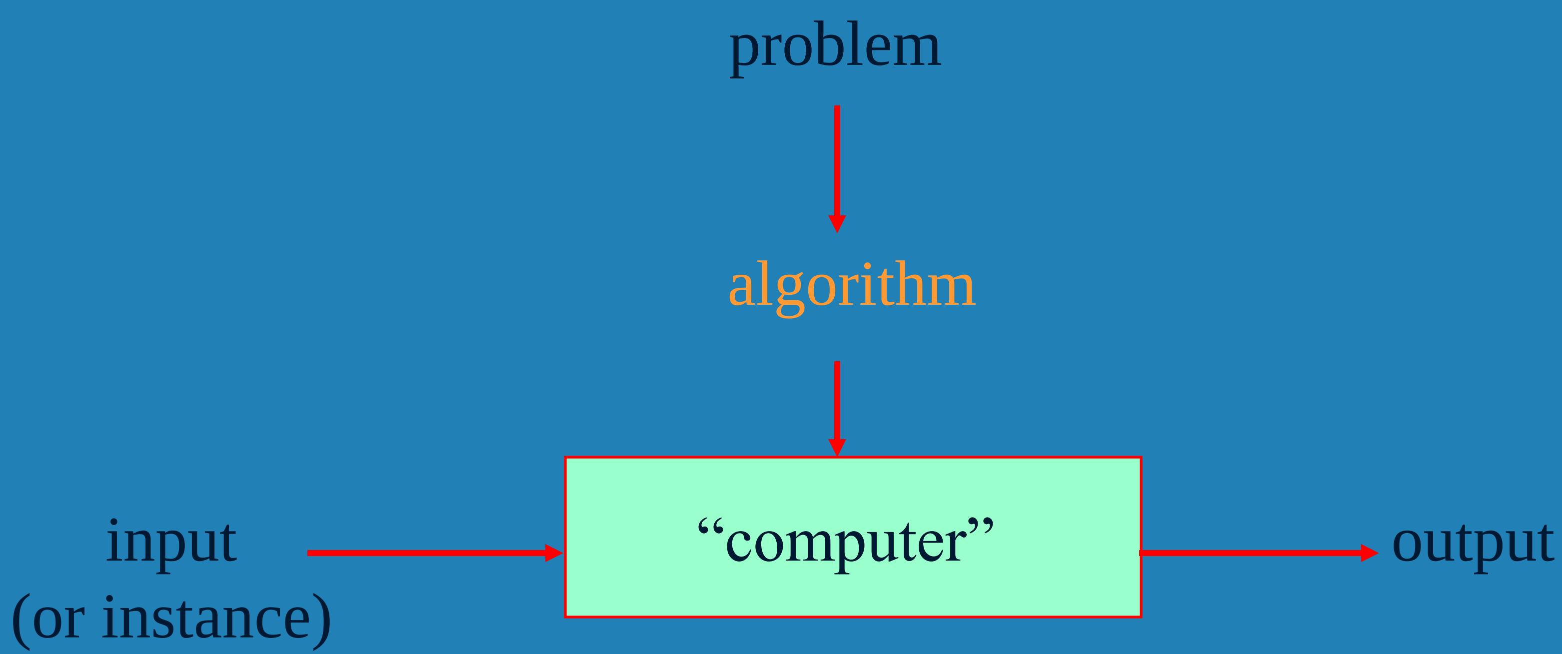
- An ***algorithm*** **is a sequence of unambiguous instructions for solving a problem, i.e., for obtaining a required output for any legitimate input in a finite amount of time.**

  - Can be represented various forms
  - Unambiguity/clearness
  - Effectiveness
  - Finiteness/termination
  - Correctness

Historical Perspective

- Euclid's algorithm for finding the greatest common divisor
- Muhammad ibn Musa al-Khwarizmi – 9th century mathematician
  www.lib.virginia.edu/science/parshall/khwariz.html

Notion of algorithm and problem



algorithmic solution
(different from a conventional solution)

Example of computational problem: sorting

- **Statement of problem:**
  - ***Input:*** **A sequence of *n* numbers** $<a_1, a_2, \ldots, a_n>$
  - ***Output:*** **A reordering of the input sequence** $<a'_1, a'_2, \ldots, a'_n>$ **so that** $a'_i \le a'_j$ **whenever** $i < j$
- **Instance: The sequence <5, 3, 2, 8, 3>**
- **Algorithms:**
  - **Selection sort**
  - **Insertion sort**
  - **Merge sort**
  - **(many others)**

Selection Sort

- **Input: array a[1], ⋯, a[n]**
- **Output: array a sorted in non-decreasing order**
- **Algorithm:**

> **for** i=1 **to** n
>
> **swap** a[i] **with smallest of** a[i], ⋯, a[n]

- Is this unambiguous? Effective?
- See also pseudocode, section 3.1

Some Well-known Computational Problems

- Sorting
- Searching
- Shortest paths in a graph
- Minimum spanning tree
- Primality testing
- Traveling salesman problem
- Knapsack problem
- Chess
- Towers of Hanoi
- Program termination

Some of these problems don't have efficient algorithms, or algorithms at all!

Basic Issues Related to Algorithms

- How to design algorithms
- How to express algorithms
- Proving correctness
- Efficiency (or complexity) analysis
  - Theoretical analysis
  - Empirical analysis
- Optimality

Algorithm design strategies

- Brute force
- Divide and conquer
- Decrease and conquer
- Transform and conquer
- Greedy approach
- Dynamic programming
- Backtracking and branch-and-bound
- Space and time tradeoffs

Analysis of Algorithms

- **How good is the algorithm?**
  - Correctness
  - Time efficiency
  - Space efficiency

- **Does there exist a better algorithm?**
  - Lower bounds
  - Optimality

What is an algorithm?

- Recipe, process, method, technique, procedure, routine,… with the following requirements:

1. Finiteness
   - terminates after a finite number of steps
2. Definiteness
   - rigorously and unambiguously specified
3. Clearly specified input
   - valid inputs are clearly specified
4. Clearly specified/expected output
   - can be proved to produce the correct output given a valid input
5. Effectiveness
   - steps are sufficiently simple and basic

Why study algorithms?

- **Theoretical importance**
  - **the core of computer science**
- **Practical importance**
  - **A practitioner's toolkit of known algorithms**
  - **Framework for designing and analyzing algorithms for new problems**

Example: Google's PageRank Technology

Euclid's Algorithm

Problem: Find gcd(m,n), the greatest common divisor of two nonnegative, not both zero integers m and n

Examples: gcd(60,24) = 12, gcd(60,0) = 60, gcd(0,0) = ?

Euclid's algorithm is based on repeated application of equality

$$\gcd(m,n) = \gcd(n, m \bmod n)$$

until the second number becomes 0, which makes the problem trivial.

Example: gcd(60,24) = gcd(24,12) = gcd(12,0) = 12

Two descriptions of Euclid's algorithm

Step 1 **If $n = 0$, return m and stop; otherwise go to Step 2**

Step 2 **Divide m by n and assign the value of the remainder to r**

Step 3 **Assign the value of n to m and the value of r to n. Go to Step 1.**

while $n \neq 0$ **do**

  $r \leftarrow m \bmod n$

  $m \leftarrow n$

  $n \leftarrow r$

return m

Other methods for computing gcd(*m*,*n*)

Consecutive integer checking algorithm

Step 1 Assign the value of min{*m*,*n*} to *t*

Step 2 Divide *m* by *t*. If the remainder is 0, go to Step 3; otherwise, go to Step 4

Step 3 Divide *n* by *t*. If the remainder is 0, return *t* and stop; otherwise, go to Step 4

Step 4 Decrease *t* by 1 and go to Step 2

Is this slower than Euclid's algorithm?
How much slower?

O(n), if n <= m , vs O(log n)

Other methods for gcd(*m*,*n*) [cont.]

Middle-school procedure

Step 1 Find the prime factorization of *m*

Step 2 Find the prime factorization of *n*

Step 3 Find all the common prime factors

Step 4 Compute the product of all the common prime factors and return it as gcd(*m*,*n*)

Is this an algorithm?

How efficient is it?

Time complexity: O(sqrt(n))

Sieve of Eratosthenes

Input: Integer $n \geq 2$

Output: List of primes less than or equal to n

for $p \leftarrow 2$ **to** n **do** $A[p] \leftarrow p$

for $p \leftarrow 2$ **to** n **do**

  if $A[p] \neq 0$ //p hasn't been previously eliminated from the list

    $j \leftarrow p * p$

    while $j \leq n$ **do**

      $A[j] \leftarrow 0$ //mark element as eliminated

      $j \leftarrow j + p$

Example: 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

Time complexity: O(n)

Two main issues related to algorithms

- How to design algorithms

- How to analyze algorithm efficiency

Algorithm design techniques/strategies

- Brute force
- Divide and conquer
- Decrease and conquer
- Transform and conquer
- Space and time tradeoffs
- Greedy approach
- Dynamic programming
- Iterative improvement
- Backtracking
- Branch and bound

Analysis of algorithms

- How good is the algorithm?
  - time efficiency
  - space efficiency
  - correctness ignored in this course

- Does there exist a better algorithm?
  - lower bounds
  - optimality

Important problem types

- sorting
- searching
- string processing
- graph problems
- combinatorial problems
- geometric problems
- numerical problems

Sorting (I)

- Rearrange the items of a given list in ascending order.
  - Input: A sequence of n numbers $<a_1, a_2, \ldots, a_n>$
  - Output: A reordering $<a'_1, a'_2, \ldots, a'_n>$ of the input sequence such that $a'_1 \leq a'_2 \leq \ldots \leq a'_n$.
- Why sorting?
  - Help searching
  - Algorithms often use sorting as a key subroutine.
- Sorting key
  - A specially chosen piece of information used to guide sorting. E.g., sort student records by names.

Sorting (II)

- Examples of sorting algorithms
  - Selection sort
  - Bubble sort
  - Insertion sort
  - Merge sort
  - Heap sort …
- Evaluate sorting algorithm complexity: the number of key comparisons.
- Two properties
  - Stability: A sorting algorithm is called stable if it preserves the relative order of any two equal elements in its input.
  - In place : A sorting algorithm is in place if it does not require extra memory, except, possibly for a few memory units.

Selection Sort

```
Algorithm SelectionSort(A[0..n-1])
//The algorithm sorts a given array by selection sort
//Input: An array A[0..n-1] of orderable elements
//Output: Array A[0..n-1] sorted in ascending order
for i ← 0 to n – 2 do
    min ← i
    for j ← i + 1 to n – 1 do
            if A[j] < A[min]
                    min ← j
    swap A[i] and A[min]
```

Searching

- Find a given value, called a search key, in a given set.
- Examples of searching algorithms
  - Sequential search
  - Binary search …

```
Input: sorted array a_i < … < a_j and key x;
m ←(i+j)/2;
while i < j and x != a_m do
    if x < a_m then j ← m-1
                    else  i ← m+1;
 if x = a_m then output a_m;
```

Time: O(log n)

String Processing

- **A string is a sequence of characters from an alphabet.**
- **Text strings: letters, numbers, and special characters.**
- **String matching: searching for a given word/pattern in a text.**

Examples:

(i) searching for a word or phrase on WWW or in a Word document

(ii) searching for a short read in the reference genomic sequence

Graph Problems

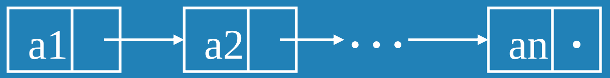
- Informal definition
  - A graph is a collection of points called vertices, some of which are connected by line segments called edges.
- Modeling real-life problems
  - Modeling WWW
  - Communication networks
  - Project scheduling …
- Examples of graph algorithms
  - Graph traversal algorithms
  - Shortest-path algorithms
  - Topological sorting

Fundamental data structures

- list
  - array
  - linked list
  - string
- stack
- queue
- priority queue/heap
- graph
- tree and binary tree
- set and dictionary

Linear Data Structures

- **Arrays**
  - **A sequence of n items of the same data type that are stored contiguously in computer memory and made accessible by specifying a value of the array's index.**
- **Linked List**
  - **A sequence of zero or more nodes each containing two kinds of information: some data and one or more links called pointers to other nodes of the linked list.**
  - **Singly linked list (next pointer)**
  - **Doubly linked list (next + previous pointers)**



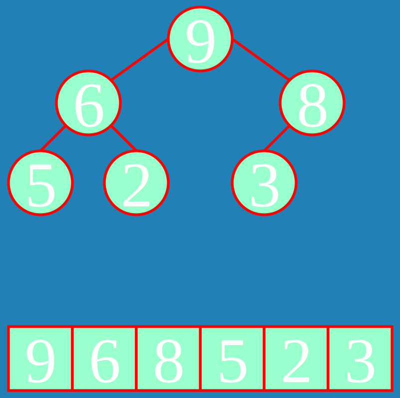
- Arrays
  - fixed length (need preliminary reservation of memory)
  - contiguous memory locations
  - direct access
  - Insert/delete
- Linked Lists
  - dynamic length
  - arbitrary memory locations
  - access by following links
  - Insert/delete

Stacks and Queues

- **Stacks**
  - A stack of plates
    - insertion/deletion can be done only at the top.
    - LIFO
  - Two operations (push and pop)
- **Queues**
  - A queue of customers waiting for services
    - Insertion/enqueue from the rear and deletion/dequeue from the front.
    - FIFO
  - Two operations (enqueue and dequeue)

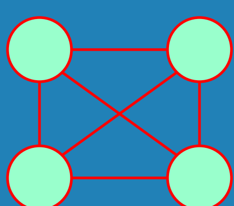
Priority Queue and Heap

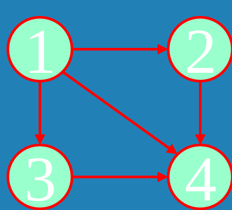
- Priority queues (implemented using heaps)
  - A data structure for maintaining a set of elements, each associated with a key/priority, with the following operations
    - Finding the element with the highest priority
    - Deleting the element with the highest priority
    - Inserting a new element
  - Scheduling jobs on a shared computer



Graphs

- Formal definition
  - A graph $G = <V, E>$ is defined by a pair of two sets: a finite set V of items called vertices and a set E of vertex pairs called edges.
- Undirected and directed graphs (digraphs).
- What's the maximum number of edges in an undirected graph with |V| vertices?
- Complete, dense, and sparse graphs
  - A graph with every pair of its vertices connected by an edge is called complete, $K_{|V|}$

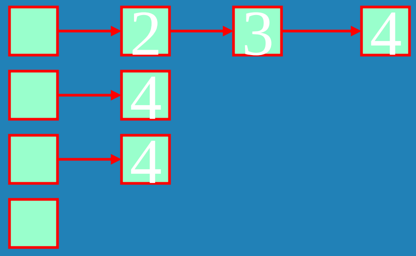




Graph Representation

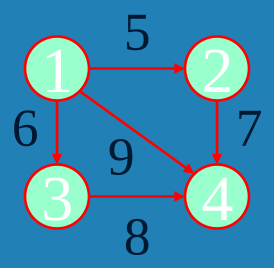
- **Adjacency matrix**
  - **n x n boolean matrix if |V| is n.**
  - **The element on the ith row and jth column is 1 if there's an edge from ith vertex to the jth vertex; otherwise 0.**
  - **The adjacency matrix of an undirected graph is symmetric.**
- **Adjacency linked lists**
  - **A collection of linked lists, one for each vertex, that contain all the vertices adjacent to the list's vertex.**
- **Which data structure would you use if the graph is a 100-node star shape?**

$$
\begin{matrix}
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{matrix}
$$



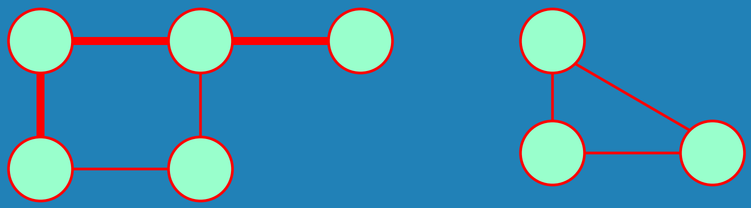
Weighted Graphs

- **Weighted graphs**
  - **Graphs or digraphs with numbers assigned to the edges.**



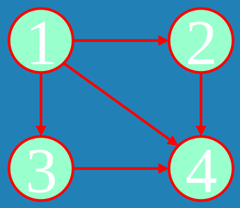
Graph Properties -- Paths and Connectivity

- **Paths**
  - A path from vertex u to v of a graph G is defined as a sequence of adjacent (connected by an edge) vertices that starts with u and ends with v.
  - Simple paths: All edges of a path are distinct.
  - Path lengths: the number of edges, or the number of vertices – 1.
- **Connected graphs**
  - A graph is said to be connected if for every pair of its vertices u and v there is a path from u to v.
- **Connected component**
  - The maximum connected subgraph of a given graph.



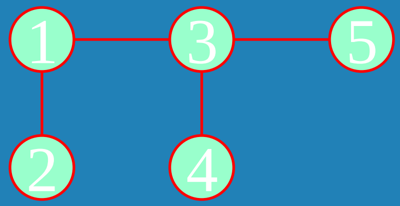
Graph Properties -- Acyclicity

- **Cycle**
  - **A simple path of a positive length that starts and ends a the same vertex.**
- **Acyclic graph**
  - **A graph without cycles**
  - **DAG (Directed Acyclic Graph)**

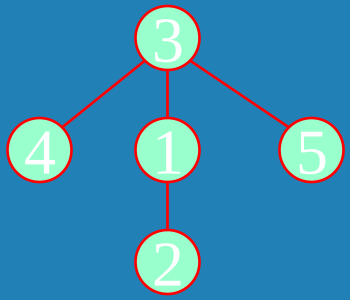


Trees

- Trees
  - A tree (or free tree) is a connected acyclic graph.
  - Forest: a graph that has no cycles but is not necessarily connected.
- Properties of trees
  - For every two vertices in a tree there always exists exactly one simple path from one of these vertices to the other. Why?
    - Rooted trees: The above property makes it possible to select an arbitrary vertex in a free tree and consider it as the root of the so called rooted tree.
    - Levels in a rooted tree.
  - $|E| = |V| - 1$



rooted

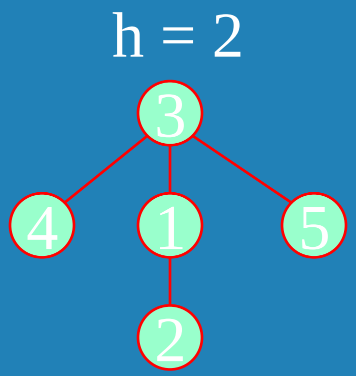


Rooted Trees (I)

- **Ancestors**
  - For any vertex *v* in a tree *T*, all the vertices on the simple path from the root to that vertex are called ancestors.
- **Descendants**
  - All the vertices for which a vertex *v* is an ancestor are said to be descendants of *v*.
- **Parent, child and siblings**
  - If *(u, v)* is the last edge of the simple path from the root to vertex *v*, *u* is said to be the parent of *v* and *v* is called a child of *u*.
  - Vertices that have the same parent are called siblings.
- **Leaves**
  - A vertex without children is called a leaf.
- **Subtree**
  - A vertex *v* with all its descendants is called the subtree of *T* rooted at *v*.

Rooted Trees (II)

- **Depth of a vertex**
  - **The length of the simple path from the root to the vertex.**
- **Height of a tree**
  - **The length of the longest simple path from the root to a leaf.**



Ordered Trees

- **Ordered trees**
  - **An ordered tree is a rooted tree in which all the children of each vertex are ordered.**
- **Binary trees**
  - **A binary tree is an ordered tree in which every vertex has no more than two children and each children is designated s either a left child or a right child of its parent.**
- **Binary search trees**
  - **Each vertex is assigned a number.**
  - **A number assigned to each parental vertex is larger than all the numbers in its left subtree and smaller than all the numbers in its right subtree.**
- **$\lfloor \log_2 n \rfloor \le h \le n - 1$, where h is the height of a binary tree and n the size.**